$$sgn\ \Pi(A - KP) = (-1)^{\alpha - \nu}\, sgn\ R\left(\frac{A}{P}\right),$$

the product extending over all primary K of degree $\alpha - \nu$.

Lemma 3. *If H runs through the primary polynomials of degree $<\nu$,*

$$sgn \prod_{H,K} (HQ - KP) = (-1)^{\rho\nu + Min.\ (\rho^2,\ \nu^2)}\, sgn \prod_{H} R\left(\frac{HQ}{P}\right).$$

7. A paper containing a detailed account of the above results, as well as a number of generalizations, has been offered to the *American Journal of Mathematics.*

* NATIONAL RESEARCH FELLOW.

[1] Cf. Dedekind, R., *J. für. Math.*, **54** (1857), pp. 1–26.

THE COLORING OF GRAPHS[1]

BY HASSLER WHITNEY

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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1. *Introduction.*—We shall give here an outline of a study of the numbers m_{ij} appearing in a formula for the number of ways of coloring a graph. The details will be given in several papers. The definitions and results in a paper on Non-separable and Planar Graphs will be made use of.

2. *The Number of Ways of Coloring a Graph.*—Suppose we assign to each vertex of a graph a color in such a way that each pair of vertices joined by an arc are of different colors. (A graph containing a 1-circuit cannot be colored therefore.) We obtain thereby a permissible coloring of the graph. Given a graph G, let m_{ij} be the number of subgraphs of rank i and nullity j. Then the number of ways of coloring G in λ colors is

$$P(\lambda) = \sum_i \lambda^{v-i} \sum_j (-1)^{i+j} m_{ij} = \sum_i m_i \lambda^{v-i},$$

if G contains v vertices. This result, first found by Birkhoff,[2] is proved by a simple logical expansion.[3]

We note that, if G contains E arcs,

$$m_{i0} + m_{i-1,\,1} + \ldots + m'_{0i} = \binom{E}{i}.$$

Let G' be formed from G by dropping out the arc ab. Let $m'_{ij}(a \times b)$ be the number of subgraphs of rank i, nullity j, of G' in which a and b are in different connected pieces. Put

$$m_i' (a \times b) = \sum_j (-1)^{i+j} m_{ij}' (a \times b).$$

Then

$$m_i = m_i' (a \times b) - m_{i-1}' (a \times b).$$

Another interesting recursion formula is as follows: m_{ij} for G equals the sum of the m_{ij}'s for all the subgraphs of G formed by dropping out a single arc, divided by $i + j$, if $i + j < E$. The first recursion formula can be extended, and gives the following results:

Let us arrange the arcs of G in a definite order: $\alpha, \beta, \gamma, \ldots, \epsilon$. Given any circuit, we form the corresponding *broken circuit* by dropping out the last arc of the circuit. Thus, if α, β, γ form a circuit, α, β is a broken circuit.

THEOREM 1. $(-1)^i m_i$ *is the number of ways of picking out i arcs from G so that not all the arcs of any broken circuit are removed.*

The coefficient of λ^v in $P(\lambda)$ is 1 if G contains no 1-circuit. We can show that the coefficient of λ^{v-R} (that is, the coefficient of λ, if G is connected), is 0. This with Theorem 1, gives

THEOREM 2.

$$(-1)^i m_i > 0, \quad i = 0, 1, \ldots, R.$$

3. *The Numbers α_i and β_i.*—Let $\alpha, \beta; \ldots; \gamma, \delta, \epsilon$ be the broken circuits of G. In theorem 1, we picked out i arcs from G so that the property A holds true, that is, α and β are not picked out, ... γ, δ and ϵ are not picked out. If we expand A by the logical expansion, α_i is the number of logical terms in the result containing i arcs. If we expand A into the second normal form (see a forthcoming paper by the author), β_i is the number of logical terms in this result containing i arcs. m_i is given in terms of the numbers α_i or β_i by certain arithmetical formulas.

4. *The Numbers m_{ij} and f_{ij}.*—To find m_{ij} for a graph G, we count all its subgraphs. But this is not necessary, for it is sufficient to count only the non-separable subgraphs.

THEOREM 3. *m_{ij} is a polynomial in the numbers N_1, N_2, ..., of non-separable subgraphs of certain types in G. The same is, therefore, true of m_i.*

For example, if G contains no 1- or 2-circuits, and N_{21}, N_{31} are the number of triangles and quadrilaterals in G, then

$$m_{31} = (E - 3)N_{21} + N_{31}.$$

Now let m_{ij} be the coefficient of $x^i y^j$ in a polynomial Q, of which the constant term is $m_{00} = 1$. Thus $Q = 1 + Q'$. Expand the logarithm of this polynomial as a power series in x and y. Let the coefficient of $x^i y^j$ in this series be f_{ij}. Then f_{ij} equals m_{ij} plus a polynomial in m_{kl}, $k \leqq i$, $l \leqq j$, containing no constant or linear term, and thus m_{ij} equals a similar expression in f_{kl}. Also, from Theorem 3 we see that f_{ij} is a polynomial in the numbers N_1, N_2, ...

Let $G = G' + G''$ be the graph containing G' and G'' as separate pieces. From the definition of m_{ij}, we find that

$$m_{ij} = \sum_{k,l} m'_{kl}\, m''_{i-k,\, j-l}.$$

THEOREM 4.

$$f_{ij} = f'_{ij} + f''_{ij}.$$

THEOREM 5. *Let $T_1, \ldots, T_q$ be types of non-separable subgraphs, and let p be any positive number. Then we can find a set of numbers $N_1, \ldots, N_q$, each $N_i > 0$, such that given any set of numbers $n_1, \ldots, n_q$, $0 \leq n_i \leq p$, there exists a graph containing $N_i + n_i$ subgraphs of type T_i, $i = 1, \ldots, q$.*

From the last two theorems we can deduce the following:

THEOREM 6. *Consider the expression for m_{ij} as a polynomial in the numbers $N_1, N_2, \ldots,$ of non-separable subgraphs of certain types of G_i. f_{ij} is exactly the linear terms of this polynomial.*

Thus if we can find the linear terms of the polynomial, we can, by a fixed arithmetical transformation, find the whole of m_{ij}.

As f_{ij} is linear, it has the form

$$f_{ij} = c_1N_1 + c_2N_2 + \ldots.$$

Is it difficult to find the coefficients $c_1, c_2, \ldots$? If $N_{k-1,1}$, is the number of k-circuits in G, the author has found the corresponding coefficients $c_{k-1,1}$ for all the numbers f_{i0}, the most interesting set. These coefficients turned out to be quite simple; it would be very worthwhile to find still more coefficients for f_{i0}.

A fundamental problem of the theory is the following: When do a set of numbers m_{ij} represent a graph? In 2 we found a relation holding between the m_{ij}.

THEOREM 7. *There exists no other polynomial relation between the m_{ij}, true for all graphs.*

However, there exist many inequalities, for instance, those induced by theorem 2. The great difficulty of the problem is now apparent.

5. *Relation to the 4-Color Map Problem.*

THEOREM 8. *Suppose G is a planar graph, and G' is its dual. Then*

$$m'_{ij} = m_{N-j,\ R-i}.$$

PROPOSITION. *Let m_{ij} be the numbers for G. If $m'_{ij} = m_{N-j,\,R-i}$ are the numbers for some graph G', then G is planar, and G' is its dual.*

This proposition, though not proved, seems very likely true. If so, we have a new statement of the 4-color map problem:

Is it true that if m_{ij} are any set of numbers such that both they and the numbers $m_{N-j,\,R-i}$ are the numbers of graphs G and G' (R and N being the rank and nullity of G), then

$$\sum_{ij} (-1)^{i+j} m_{ij} 4^{v-i} > 0?$$

More generally, this study is meant to throw some light on the nature of the numbers m_{ij}, which may in time lead to solutions of problems such as the 4-color problem.

[1] Presented to the American Mathematical Society, Oct. 25, 1930.

[2] Birkhoff, G. D., *Ann. Math.*, 2, **14**, No. 1 (42–46).

[3] See Whitney, H., *Bull. Am. Math. Soc.*, Abstract No. 36–11–396.

NON-SEPARABLE AND PLANAR GRAPHS[1]

BY HASSLER WHITNEY

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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1. *Introduction.*—We shall give here an outline of the main results of a research on non-separable and planar graphs. The methods used are entirely of a combinatorial character; the concepts of rank and nullity play a fundamental rôle. The results will be given in detail in a later paper.

A *graph* G is composed of two sets of symbols: *vertices*, a, b, $\ldots$, f, and *arcs*, $\alpha(ab)$ (or simply ab), $\beta(ac)$, $\ldots$, $\delta(ef)$. A *chain* is a set of distinct arcs and vertices, ab, bc, $\ldots$, de. A *suspended chain* is a chain containing at least two arcs, of which no vertices are on other arcs but the first and last vertices, which are each on at least two other arcs. A *circuit* is a set of distinct arcs and vertices, ab, bc, $\ldots$, de, ea. A *k-circuit* is a circuit containing k arcs. A *subgraph* H of G is a graph formed by dropping out arcs from G. Let V, E, P be the number of vertices, arcs and connected pieces in G. We define the *rank* R and the *nullity* (or cyclomatic number) N by the equations

$$R = V - P,$$
$$N = E - R = E - V + P.$$

2. *Non-Separable Graphs.*—G is called *non-separable* if it is connected, and if there are no two graphs G_1 and G_2, each containing at least one arc, which form G if a vertex of one is made to coalesce with a vertex of the other. If G is not non-separable, it is separable. G is called *cyclicly connected* if each pair of vertices is contained in a circuit in G.

THEOREM 1. *Let G be a graph containing at least two arcs but no 1-circuit. A necessary and sufficient condition that G be non-separable is that G be cyclicly connected.*[2]

Suppose a connected part of G is separable. We then separate it at a